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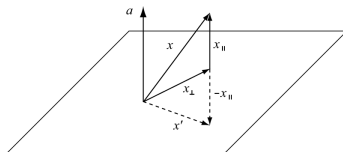
A Clifford perspective on reflection groups, root systems and ADE correspondences

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Reflections in Geometric Algebra



Reflections

$$x = x_{\perp} + x_{\parallel} \rightarrow x' = x_{\perp} - x_{\parallel} = x - 2x_{\parallel} = x - 2(x \cdot n)n$$

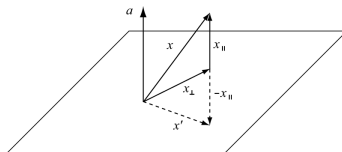
Vector space with an inner product

Why not work with the Clifford algebra?

Geometric product $ab \equiv a \cdot b + a \wedge b$

Inner product is the symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$

Reflections in Geometric Algebra



Reflections

$$x = x_{\perp} + x_{\parallel} \rightarrow x' = x_{\perp} - x_{\parallel} = x - 2x_{\parallel} = x - 2(x \cdot n)n = -nxn$$

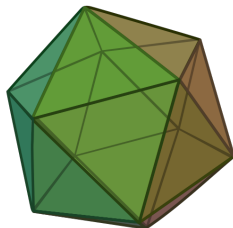
Vector space with an inner product

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Groups of reflections (Coxeter groups)



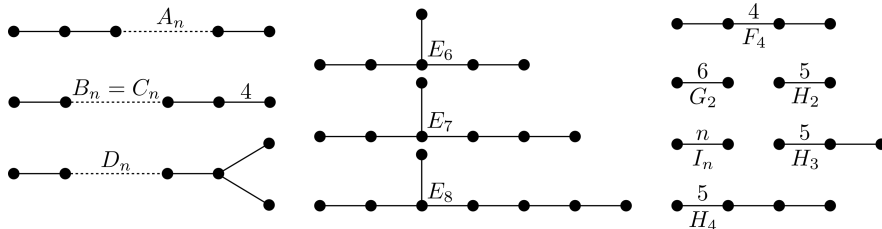
Reflection groups from generating reflections

$$x' = -nxn \rightarrow x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 =: \pm A x \tilde{A}$$

Cartan-Dieudonné theorem

Any orthogonal transformation can be written as the product of successive reflections.

Classification of Euclidean reflection groups



Types

crystallographic (Weyl/Lie theory, A-G) vs non-crystallographic (I & H), simply-laced (ADE) etc

Groups and double covers

$$x' = -nxn = -(-n)x(-n)$$

So n and $-n$ **doubly cover** the same reflection.

Reflection and rotation groups

Reflection groups: subgroups of $O(n) \Rightarrow$ doubly covered in $\text{Pin}(n)$

Rotation subgroups: subgroups of $SO(n) \Rightarrow$ doubly covered in $\text{Spin}(n)$

Icosahedral groups

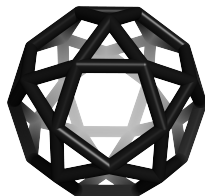
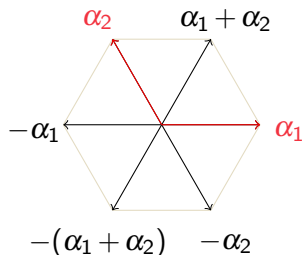
Reflection/Coxeter group H_3 in $O(3)$ of order 120.

Rotational subgroup A_5 in $SO(3)$ of order 60.

Pin group of H_3 in $\text{Pin}(3)$ of order 240.

Binary icosahedral group $2I$ in $\text{Spin}(3)$ of order 120 (or $SU(2)$ or quaternions).

Root systems



Root system Φ

A set of vectors α in a **vector space** with an **inner product** such that

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
2. $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

where the **reflections** and **Coxeter element** are $s_\alpha : v \rightarrow s_\alpha(v) = -\alpha v \alpha$ and

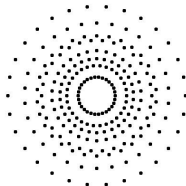
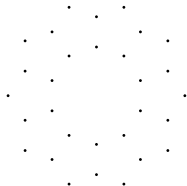
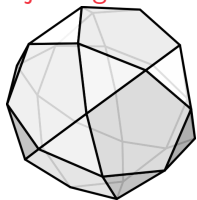
$$W = s_1 \dots s_n$$

Vector space with an inner product

Why not work with the Clifford algebra?

The Coxeter Plane

- **Every** (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by **projecting** their root system onto the Coxeter plane



Platonic Solids



A_1^3

A_3

B_3

H_3

Platonic Solid	Group
Tetrahedron	A_3 A_1^3
Octahedron Cube	B_3
Icosahedron Dodecahedron	H_3

- Platonic Solids have been known for millennia
- Described by Coxeter groups

Clifford Algebra of 3D: the relation with 4D and 8D

- Clifford (Pauli) algebra in 3D is

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1 e_2 e_3\}}_{1 \text{ trivector}}$$

- We can multiply together root vectors in this algebra $\alpha_i \alpha_j \dots$
- A general element has 8 components: 8D
- even products (rotations/spinors) have four components:

$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow R \tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

- So behaves as a 4D Euclidean object – inner product

$$(R_1, R_2) = \frac{1}{2}(R_2 \tilde{R}_1 + R_1 \tilde{R}_2)$$

Platonic Solids



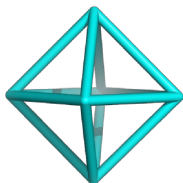
A_1^3	A_1^4
A_3	D_4
B_3	F_4
H_3	H_4

- **Abundance** of 4D root systems – **exceptional**
- Concatenating 3D reflections gives 4D **Clifford** spinors (**binary polyhedral groups**)
- These **induce 4D root systems**

$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow$$

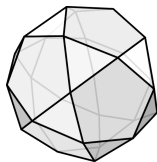
$$R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$
- This construction accidental to 3D perhaps explains the unusual abundance of 4D root systems

Spinors from reflections: easy example



- The 6 **roots** $(\pm 1, 0, 0)$ and permutations in $A_1 \times A_1 \times A_1$
- $\boxed{\pm e_1, \pm e_2, \pm e_3}$ generate **group of 8 spinors**
 $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$
- This is a **discrete spinor group** isomorphic to the **quaternion** group Q .
- As 4D vectors these are $(\pm 1, 0, 0, 0)$ and permutations, the 8 **roots** of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

Spinors from reflections: icosahedral case



- The H_3 root system has 30 **roots** e.g. simple roots

$$\alpha_1 = e_2, \alpha_2 = -\frac{1}{2}((\tau-1)e_1 + e_2 + \tau e_3) \text{ and } \alpha_3 = e_3.$$

- Subgroup of **rotations** A_5 of order **60** is doubly covered by **120**

spinors of the form $\alpha_1 \alpha_2 = -\frac{1}{2}(1 - (\tau-1)e_1 e_2 + \tau e_2 e_3),$

$\alpha_1 \alpha_3 = e_2 e_3$ and $\alpha_2 \alpha_3 = -\frac{1}{2}(\tau - (\tau-1)e_3 e_1 + e_2 e_3).$

- These 120 spinors/roots constitute the H_4 root system

3D to 4D Induction (and 2D to 2D)

Induction theorem (3D to 4D)

The 3D root systems $(A_1 \times I_2(n), A_3, B_3, H_3)$ yield the 4D root systems $(I_2(n) \times I_2(n), D_4, F_4, H_4)$

Induction theorem (2D to 2D)

The 2D root systems $I_2(n)$ are self-dual i.e. only yield themselves $I_2(n)$

Arnold's Trinities

Arnold's observation that many areas of real mathematics can be **complexified** and **quaternionified** resulting in theories with a similar structure.

- The **fundamental trinity** is thus $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The **projective spaces** $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The **spheres** $(\mathbb{R}P^1 = S^1, \mathbb{C}P^1 = S^2, \mathbb{H}P^1 = S^4)$
- The **Möbius/Hopf bundles** $(S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)$
- The **Lie Algebras** (E_6, E_7, E_8)
- The symmetries of the **Platonic Solids** (A_3, B_3, H_3)
- The **4D groups** (D_4, F_4, H_4)
- **New connections** via this **Clifford spinor construction**

Platonic Trinities

- Arnold's connection between (A_3, B_3, H_3) and (D_4, F_4, H_4) is **very convoluted** and indirect
- **Decomposition of the projective plane** into Weyl chambers and Springer cones
- The **number of Weyl chambers** in each segment is $24 = 2(1 + 3 + 3 + 5)$, $48 = 2(1 + 5 + 7 + 11)$, $120 = 2(1 + 11 + 19 + 29)$
- Notice this miraculously **is one less than the degrees of polynomial invariants** $((2, 4, 4, 6), (2, 6, 8, 12), (2, 12, 20, 30))$ of the Coxeter groups (D_4, F_4, H_4)
- The Clifford connection is **much more direct**
- But worth looking at these **invariants** as a clue

Trinity correspondence

- We now know the binary icosahedral group is essentially H_4 , octahedral F_4 and tetrahedral D_4
- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras ($A_{2n-1}, D_{n+2}, E_6, E_7, E_8$)
- $(2n+2, 12, 18, 30)$ are the sum of the dimensions of the irreps and the ADE Coxeter number

2D/3D	$ \Phi $	4D	G	$\sum d_i$	ADE	h
invariants						
A_3	12	D_4	$2T$	12		
B_3	18	F_4	$2O$	18		
H_3	30	H_4	$2I$	30		

More than a Trinity

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- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras ($A_{2n-1}, D_{n+2}, E_6, E_7, E_8$)
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2D/3D	$ \Phi $	4D	G	$\sum d_i$	ADE	h
Direct induction						
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$		
A_3	12	D_4	$2T$	12		
B_3	18	F_4	$2O$	18		
H_3	30	H_4	$2I$	30		

But less than a McKay/ADE correspondence

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- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras ($A_{2n-1}, D_{n+2}, E_6, E_7, E_8$)
- $(2n+2, 12, 18, 30)$ are the sum of the dimensions of the irreps and the ADE Coxeter number

	4D	G	$\sum d_i$	ADE	h
	McKay/ADE			\tilde{A}_{2n-1}	$2n$
	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$	\tilde{D}_{n+2}	$2(n+1)$
	D_4	$2T$	12	\tilde{E}_6	12
	F_4	$2O$	18	\tilde{E}_7	18
	H_4	$2I$	30	\tilde{E}_8	30

The countably infinite family $A_1 \times I_2(n)$ and Arnold's construction

For A_1^3

Can see immediately $8 = 2(1 + 1 + 1 + 1)$

For A_1^4

Simple roots $\alpha_1 = e_1, \alpha_2 = e_2, \alpha_3 = e_3, \alpha_4 = e_4$ give

$$W = e_1 e_2 e_3 e_4 = (\cos \frac{\pi}{2} + \sin \frac{\pi}{2} e_1 e_2)(\cos \frac{\pi}{2} + \sin \frac{\pi}{2} e_3 e_4) = \exp(\frac{\pi}{2} e_1 e_2) \exp(\frac{\pi}{2} e_3 e_4)$$

Gives exponents $(1, 1, 1, 1)$ (from $h - 1 = 2 - 1$)

The countably infinite family $A_1 \times I_2(n)$ and Arnold's construction

For $A_1 \times I_2(n)$

One gets the same decomposition $4n = 2(1 + (n-1) + 1 + (n-1))$

For $I_2(n) \times I_2(n)$

Simple roots $\alpha_1 = e_1$, $\alpha_2 = -\cos \frac{\pi}{n} e_1 + \sin \frac{\pi}{n} e_2$, $\alpha_3 = e_3$,
 $\alpha_4 = -\cos \frac{\pi}{n} e_3 + \sin \frac{\pi}{n} e_4$ give $W = \exp\left(-\frac{\pi e_1 e_2}{n}\right) \exp\left(-\frac{\pi e_3 e_4}{n}\right)$
 Gives exponents $(1, (n-1), 1, (n-1))$

More than a Trinity

- We now know the binary icosahedral group is essentially H_4 , octahedral F_4 and tetrahedral D_4
- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras ($A_{2n-1}, D_{n+2}, E_6, E_7, E_8$)
- $(2n+2, 12, 18, 30)$ are the sum of the dimensions of the irreps and the ADE Coxeter number

2D/3D	$ \Phi $	4D	G	$\sum d_i$	ADE	h
Direct induction						
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$		
A_3	12	D_4	$2T$	12		
B_3	18	F_4	$2O$	18		
H_3	30	H_4	$2I$	30		

The countably infinite family $I_2(n)$ and Arnold's construction

For $I_2(n)$

one gets the same decomposition $2n = 2(1 + (n-1))$

For $I_2(n)$

simple roots $\alpha_1 = e_1$, $\alpha_2 = -\cos \frac{\pi}{n} e_1 + \sin \frac{\pi}{n} e_2$ give
 $W = \exp\left(-\frac{\pi e_1 e_2}{n}\right)$ gives exponents $(1, (n-1))$

The countably infinite families and Arnold's construction

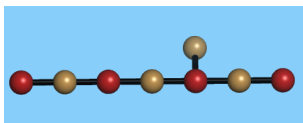
- So Arnold's initial hunch regarding the exponents **extends in fact to my full correspondence**
- **McKay correspondence** is a correspondence between even subgroups of $SU(2)/\text{quaternions}$ and ADE affine Lie algebras
- In fact here get the even quaternion subgroups from 3D – **link to ADE affine Lie algebras** via McKay?

2D/3D, 2D/4D and ADE correspondences

- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras ($A_{2n-1}, D_{n+2}, E_6, E_7, E_8$)
- Induction theorem: get these as 2D/4D root systems ($I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4$) from 2D/3D root systems ($I_2(n), A_1 \times I_2(n), A_3, B_3, H_3$)
- $(2n, 2n+2, 12, 18, 30)$ are numbers of roots, the sum of the dimensions of the irreps and the ADE Coxeter number

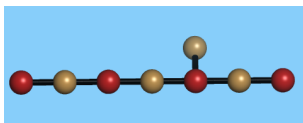
2D/3D	$ \Phi $	4D	G	$\sum d_i$	ADE	h
$I_2(n)$	$2n$	$I_2(n)$	C_{2n}	$2n$	\tilde{A}_{2n-1}	$2n$
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$	\tilde{D}_{n+2}	$2(n+1)$
A_3	12	D_4	$2T$	12	\tilde{E}_6	12
B_3	18	F_4	$2O$	18	\tilde{E}_7	18
H_3	30	H_4	$2I$	30	\tilde{E}_8	30

Is there a direct Platonic-ADE correspondence?



2D/3D		rot	ADE		legs
$I_2(n)$		n			
$A_1 \times I_2(n)$		$2, 2, n$			
A_3		$2, 3, 3$	E_6		$2, 3, 3$
B_3		$2, 3, 4$	E_7		$2, 3, 4$
H_3		$2, 3, 5$	E_8		$2, 3, 5$

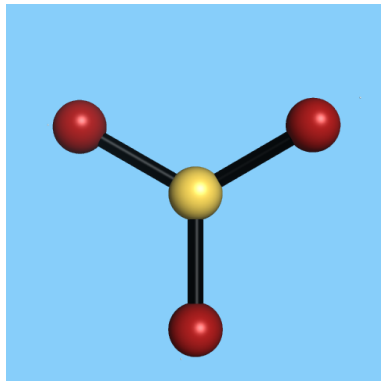
Is there a direct Platonic-ADE correspondence?



2D/3D		rot	ADE		legs
$I_2(n)$		n	A_n		n
$A_1 \times I_2(n)$		$2, 2, n$	D_{n+2}		$2, 2, n$
A_3		$2, 3, 3$	E_6		$2, 3, 3$
B_3		$2, 3, 4$	E_7		$2, 3, 4$
H_3		$2, 3, 5$	E_8		$2, 3, 5$

A Trinity of root system ADE correspondences

- 2D/3D root systems ($I_2(n), A_1 \times I_2(n), A_3, B_3, H_3$)
- 2D/4D root systems ($I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4$)
- ADE root systems ($A_n, D_{n+2}, E_6, E_7, E_8$)

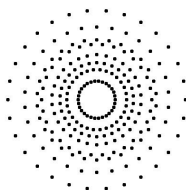
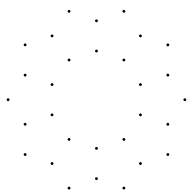
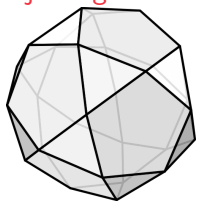


Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have **invariant polynomials**. Their **degrees** d are important invariants/group characteristics.
- E.g. each one has the Euclidean distance polynomial $x_1^2 + \cdots + x_n^2$ of degree 2
- Turns out that actually **degrees** d are intimately related to so-called **exponents** m $m = d - 1$, related to complex eigenvalues of the **Coxeter element**.

The Coxeter Plane

- **Every** (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by **projecting** their root system onto the Coxeter plane



Arnold's indirect connection between Trinities (A_3, B_3, H_3) and (D_4, F_4, H_4)

- **Arnold** had noticed a handwavey connection:
- Decomposition of 3D groups in terms of number of **Springer cones** matches what are essentially the **exponents** of the 4D groups:
- A_3 : $24 = 2(1 + 3 + 3 + 5) - D_4$: $(1, 3, 3, 5)$
- B_3 : $48 = 2(1 + 5 + 7 + 11) - F_4$: $(1, 5, 7, 11)$
- H_3 : $120 = 2(1 + 11 + 19 + 29) - H_4$: $(1, 11, 19, 29)$

Coxeter Elements, Degrees and Exponents

Standard exposition

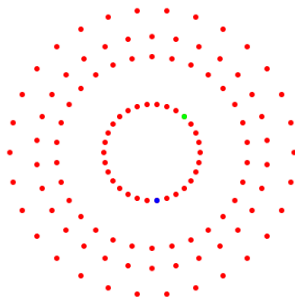
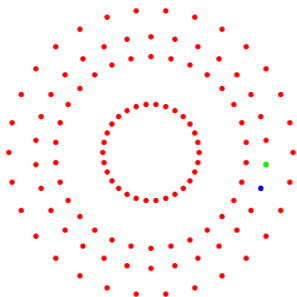
“In order to bring the eigenvalues of the Coxeter element w into the picture, we have to complexify the situation”.

- The Coxeter element has **complex eigenvalues** of the form $\exp(2\pi mi/h)$ where m are called **exponents**
- Standard theory **complexifies** the real Coxeter group situation in order to find **complex eigenvalues**, then takes **real** sections again.
- In particular, **1** and **$h-1$** are always exponents
- Turns out that actually **exponents and degrees** are intimately related (**$m = d-1$**). The construction is slightly roundabout but uniform, and uses the **Coxeter plane**.

4D case: H_4

- E.g. H_4 has exponents 1, 11, 19, 29
- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$$



Arnold's indirect connection between Trinities

rank 4	exponents	W-factorisation
D_4	1, 3, 3, 5	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} I B_C\right)$
F_4	1, 5, 7, 11	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} I B_C\right)$
H_4	1, 11, 19, 29	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$

The **remaining cases** in the root system induction construction work the same way, not just this Trinity! So more general correspondence:

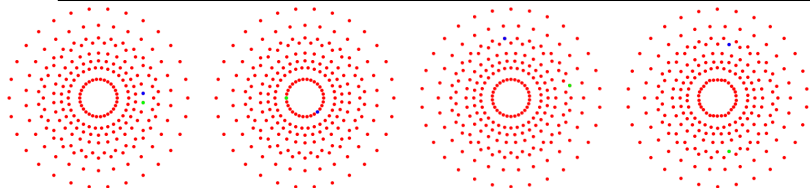
$$(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3) \rightarrow (I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$$

8D case: E_8

- E.g. H_4 has exponents 1,11,19,29, E_8 has 1,7,11,13,17,19,23,29

- Coxeter versor decomposes into **orthogonal components**

$$W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{7\pi}{30} B_2\right) \exp\left(\frac{11\pi}{30} B_3\right) \exp\left(\frac{13\pi}{30} B_4\right)$$



Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can **factorise** W into **orthogonal** (commuting/anticommuting) components

$$W = \alpha_1 \dots \alpha_n = W_1 \dots W_n \text{ with } W_i = \exp(\pi m_i l_i / h)$$
- Here, l_i is a bivector describing a **plane** with $l_i^2 = -1$
- For v **orthogonal to the plane** described by l_i we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v \text{ so cancels out}$$
- For v **in the plane** we have

$$v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i l_i / h) v$$
- Thus if we **decompose** W into **orthogonal eigenspaces**, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

Conclusions

- Clifford algebra provides a very **general** way of doing (reflection) **group** theory (Cartan-Dieudonné)
- Construction of the **exceptional root systems** from 3D root systems
- New and streamlined **connections** between algebraic concepts with increased understanding of the **geometry**
- More **geometric** approach to the geometry of the **Coxeter plane, degrees and exponents**
- **Conceptual unification** by working at the level of **root systems** (not group theory, Lie theory, polytopes etc)

Conclusions

Thank you!

Some papers with further details

- Dechant P-P, From the Trinity (A_3, B_3, H_3) to an ADE correspondence, Proceedings of the Royal Society A 474 (2220), 20180034
- Dechant P-P, The birth of E_8 out of the spinors of the icosahedron, PRSA 472(2185):20150504
- Dechant P-P, Clifford algebra is the natural framework for root systems and Coxeter groups. Group theory: Coxeter, conformal and modular groups. AACA 2017 1;27(1):17-31
- Dechant P-P, Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups. AACA 2013 1;23(2):301-21.